

Revisiting Mean-Variance Optimization

From a scenario analysis perspective.

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We have used a mean-variance optimization model in the management of our bond portfolios for more than a decade. One of the optimizer's enduring attributes has been its willingness to take large overweights relative to our performance benchmarks in non-Treasury sectors of the bond market.

The reason for this is twofold. First, the relative yield advantage of corporates, mortgages, and other "spread" sectors far exceeds those sectors' default and liquidity costs, and the optimizer likes the extra yield. Second, the model does not recognize any special risk resulting from not being invested in Treasuries, apart from the performance risk relative to the benchmark related to spread volatility. Consequently, the optimizer often calls for a zero weight in Treasuries, particularly when spreads reach the levels experienced in 2000.

Several of our colleagues have expressed concern with this result and, more generally, with the validity of the optimization process, which seems so willing to eliminate Treasuries from the portfolio across a wide range of spreads. It is also our experience that most bond portfolio managers use scenario analysis rather than optimization to construct portfolios. The appeal of scenario analysis is its formal recognition of the key risks that we face in pursuing any particular sector, maturity distribution, or duration strategy.

Given the concerns about our process and the popularity of the principal alternative, scenario analysis, we reexamine our approach here. We illustrate in some detail

the basic problem with the mean-variance optimization framework for bond portfolios, while showing how to integrate scenario analysis into the optimization framework as an intuitive way to consider non-normal returns. We also examine the impact of specification of the utility function on portfolio choice.

LAYING THE FOUNDATION

Nearly fifty years ago, Harry Markowitz [1952] had a great insight that revolutionized portfolio construction—that the “optimal” portfolio could be found by maximizing the expected return minus some multiple of variance. This breakthrough made it possible to achieve an optimal portfolio with the limited computing power available at the time.

Over the years, the mean-variance approach has come to dominate the portfolio selection process, at least academically, but it is often forgotten that the original problem is how to maximize the expected value of the investor’s utility. Mean-variance is a shortcut that solves it perfectly only under a restrictive set of assumptions. With today’s computing power, it is possible to relax the restrictions and solve the underlying problem directly.

To reexamine mean-variance optimization, it helps to simplify the market. We have aggregated the mortgage and agency sectors, and assumed that the market comprises 25% Treasuries, 50% mortgages/agencies, and 25% corporates. We have also assumed that all bonds have durations of five years, and we have ignored convexity, term structure effects, defaults, and transaction costs. Thus, the only decision the optimizer must make is one of sector allocation. None of our simplifications is material to the points that we are trying to make, which are equally valid in a more realistic setting.

Let’s assume that spreads are not expected to change and that our expected returns are therefore equal to the average yields in each sector. As a result, corporates and mortgages/agencies are expected to outperform Treasuries by their yield spreads.¹

Risk is defined to be tracking error, or the performance risk arising from deviations from the specified benchmark. The focus on tracking error rather than absolute risk is typical for mandates that most investment managers receive. The optimization process finds the portfolio that maximizes the return on the portfolio minus a penalty for its risk [based on a measure of our risk aversion— λ in Equation (1)].

Specifically:

$$\text{Maximize } E(r) - \lambda\sigma^2 \quad (1)$$

We use $\lambda = 25$.²

What will an optimizer do with this input? Given current spread levels in the U.S. fixed-income markets and the statistics shown in Exhibit 1, the optimizer puts 100% of the portfolio into corporates and nothing into Treasuries, mortgages, and agencies. Clearly, 170 basis points is too good to turn down.

The resulting portfolio statistics vis-à-vis our benchmark are in Exhibit 2. The 85 basis point premium is the portfolio’s expected return (which we have assumed to be equal to its weighted-average yield) minus the expected return of the market index. The expected tracking error of 104 basis points represents the standard deviation of the expected premium, given the portfolio’s sector weights versus those of the index, the volatility of spreads, and the correlation of spreads with one another.

To give context to the extent of the expected tracking error, the median standard deviation of the excess return on a sample of active money managers benchmarked against the Lehman Brothers Aggregate Index has been 70 basis points over the five years ending September 30, 2000; 105 basis points marks the 25th percentile.³ Thus, in our example, the optimizer is taking a fairly large bet.

We discuss the utility calculation in considerable detail later. For now, it is helpful to think of it as the quantification of the value to us (in basis points) of the return/risk combination.

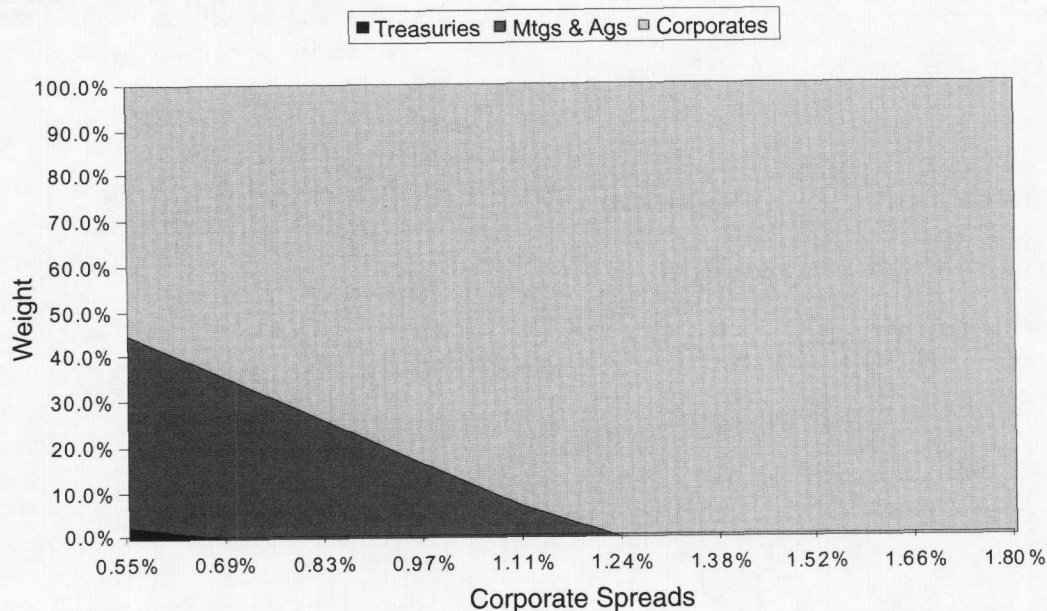
EXHIBIT 1 OPTIMIZATION PARAMETERS

	Spreads versus Treasuries	Volatility of Spreads	Correlation of Spreads
Mortgages and Agencies	0.85%	0.31%	0.8
Corporates	1.70%	0.42%	

EXHIBIT 2 PORTFOLIO RISK AND RETURN STATISTICS

Expected Premium	0.85%
Expected Tracking Error	1.04%
Utility	0.52%

EXHIBIT 3 SECTOR WEIGHTS ACROSS SPREADS



The allocation of the entire portfolio to corporates may be reasonable, considering that current corporate spreads are near record levels. As you can see in Exhibit 3, however, even when we consider narrower spreads, corporates are substantially overweighted, and the Treasury weighting is minimal at most. This strong bias to corporates in particular and to “spread product” in general is the source of the concern we note above, and the motivation for reevaluating whether we need to modify our optimization process.⁴

The most obvious suspects are misstatements of the expected returns, risks, or appropriate level of risk aversion. Yet, even when we make quite large adjustments to those factors, the conclusions remain unchanged. For example, default costs, which we have ignored, would have to be more than ten times their normal value to reduce the corporate weighting.⁵ Even then, the optimizer would replace corporates with mortgages and agencies rather than Treasuries. Similarly, you would have to assume spread volatility substantially above historical levels to alter this picture significantly.

As for risk aversion, conventional values of λ for asset allocation range between 0.5 and 10. We are assuming more risk aversion than that (as is appropriate in an only fixed-income context). We would need to double our risk aversion to change this result. In addition, from many years of experience, we have found that this value of λ does a reasonable job in other fixed-income trade-offs.⁶

SETTING THE SCENARIOS

Although a mean-variance optimization approach is taken for granted in many finance textbooks, scenario analysis is the principal decision-making tool for most bond managers. Given its appeal, we undertook to integrate scenario analysis into our optimization approach to see if it makes a difference.

We could consider any number of scenarios, but for the sake of simplicity we limit ourselves to three. The first scenario is our best guess for the future. Let's assume that this is a steady-spread scenario, with the same volatility and correlation parameters as in Exhibit 1. As a second scenario, we envision a substantial widening in spreads, and as a third scenario, we picture a more modest spread narrowing.⁷

Exhibit 4 describes these three possibilities.

Continuing to ignore any convexity or term structure effects, the expected return premium is simply the spreads from Exhibit 1 less the change in spreads multiplied by the duration, which we have assumed to be five years. We show the results in Exhibit 5.

How do we use this information to construct a portfolio? One approach is to run the optimizer for each scenario, and then average the optimal holdings produced under each. In the steady-spread and narrowing spread scenarios, the optimizer puts the entire portfolio into corporates. Given the degree of widening that we have

**EXHIBIT 4
SCENARIO SPREADS TO TREASURIES**

	Wider	Steady	Narrower
Mortgages and Agencies	1.35%	0.85%	0.60%
Corporates	2.70%	1.70%	1.20%

**EXHIBIT 5
SCENARIO RETURNS VERSUS TREASURIES**

	Wider	Steady	Narrower
Treasuries	0.00%	0.00%	0.00%
Mortgages and Agencies	-1.65%	0.85%	2.10%
Corporates	-3.30%	1.70%	4.20%
Index	-1.65%	0.85%	2.10%

**EXHIBIT 6
SECTOR WEIGHTS IN EACH SCENARIO**

	Wider	Steady	Narrower	Weighted Average
Treasuries	100%	0%	0%	10%
Mortgages and Agencies	0%	0%	0%	0%
Corporates	0%	100%	100%	90%
Scenario Probability	10%	70%	20%	

**EXHIBIT 7
PORTFOLIO RISK AND RETURN STATISTICS**

	Average of Three Scenarios	Initial Optimization
Expected Premium	0.68%	0.85%
Expected Risk	0.85%	1.04%
Utility	0.43%	0.52%

assumed for the remaining scenario, it is not surprising that the optimizer calls for a Treasury-only portfolio in this scenario.

Assuming the probabilities shown in Exhibit 6, the weighted-average portfolio would have 10% in Treasuries and 90% in corporates.

If our objective were to get Treasuries into the portfolio, this has accomplished our goal. But at what cost? The introduction of Treasuries has reduced the portfolio's expected premium by 17 basis points and its expected risk by 19 basis points (Exhibit 7). The overall utility of the new portfolio is significantly lower, telling us that this crude solution is coming at considerable expense in terms of portfolio efficiency.

This doesn't mean that we need to give up on scenario analysis, but it does mean that we need to consider a more complicated optimization process.

SCENARIO OPTIMIZATION

Since Markowitz [1952], traditional optimization has considered only the mean and variance of portfolios.⁸ These two statistics are presumed to be sufficient to characterize the investor's preferences across portfolios. This is true only in the special case in which returns are *normally distributed* and the utility function is of the form:

$$u(r) = \frac{e^{-2\lambda r} - 1}{-2\lambda} \quad (2)$$

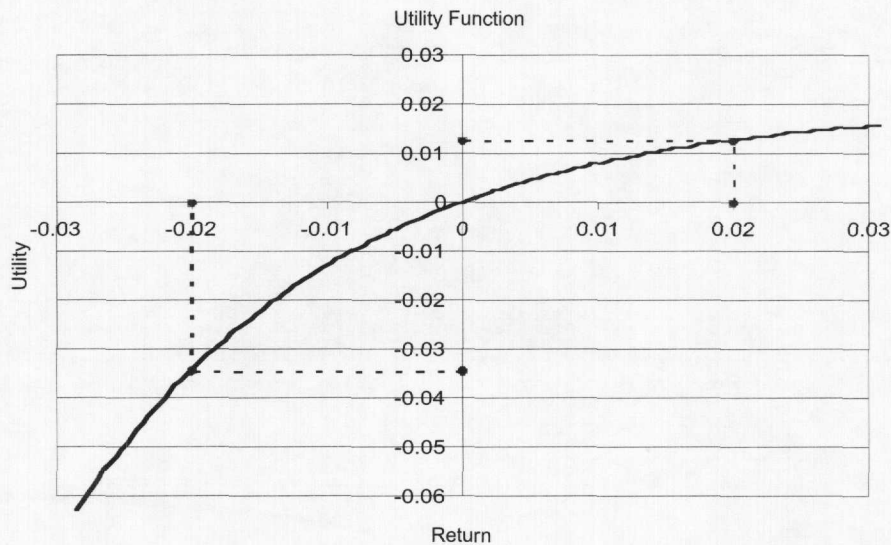
where $u(r)$ is the utility of a return r (where return is in fact excess return relative to the bond market), and λ is a constant reflecting our degree of risk aversion.⁹

λ must be positive in order for the utility function to be concave—that is, to reflect the investor's risk aversion by having losses count more than gains of the same size. The larger it is, the more concave (and therefore risk-averse) the function is. A λ of 0.5 maximizes long-term growth but reflects very little regard for risk. λ s of 2 to 10 are common for asset allocation applications. We have used λ s ranging from 15 to 30 for intermediate fixed-income clients depending on their specific objectives.

For a λ of 25, the utility function looks like the graph in Exhibit 8. This utility function states that a bond that matches the return of the market has a utility of zero, while a bond that outperforms has a positive utility, and one that underperforms has a negative utility. Specifically, a bond with a positive excess return of 2% would have a utility of 1.3, while a bond with an excess return of -2% would have a utility of roughly -3.4. If these are the only two possible outcomes and they are equally likely, the expected utility would be -1.1, the weighted average of 1.3 and -3.4.

Of course, this is only a two-scenario example, and reality is much more complex. The process for more complex cases, however, is identical in concept. For each

EXHIBIT 8
UTILITY FUNCTION



possible outcome, we take the probability of that outcome times its utility, summing the results for all the possible outcomes. For continuous probability distributions (with an infinite number of possibilities) we average small intervals. These intervals can be made as small as necessary to provide a sufficient degree of accuracy.

Mathematically, we can describe this as:

$$E(u) = \sum u(r_i) \text{Probability}(r_i) \quad (3)$$

or

$$E(u) = \int u(r)pdf(r)dr \quad (4)$$

in continuous form, where $pdf(r)$ is the probability distribution of the expected returns.

The goal of optimization is to find the portfolio combination that has the highest utility. The process begins by assuming a set of weights that produce a portfolio (π) and calculating the expected return and risk of this portfolio as follows:

$$\hat{r}_\pi = \sum_i \omega_i \hat{r}_i \quad (5)$$

and

$$\hat{\sigma}_\pi^2 = \sum_i \sum_j \rho_{i,j} \hat{\sigma}_i \hat{\sigma}_j \omega_i \omega_j \quad (6)$$

where ω are the sector weights, σ is the variability of sector relative returns, and ρ is the correlation of the sectors' spread changes. As we've already seen, the portfolio's expected return can be divided into many small pieces, each with an associated probability. We calculate the utility of the total by multiplying the probability of each discrete return by its associated utility.

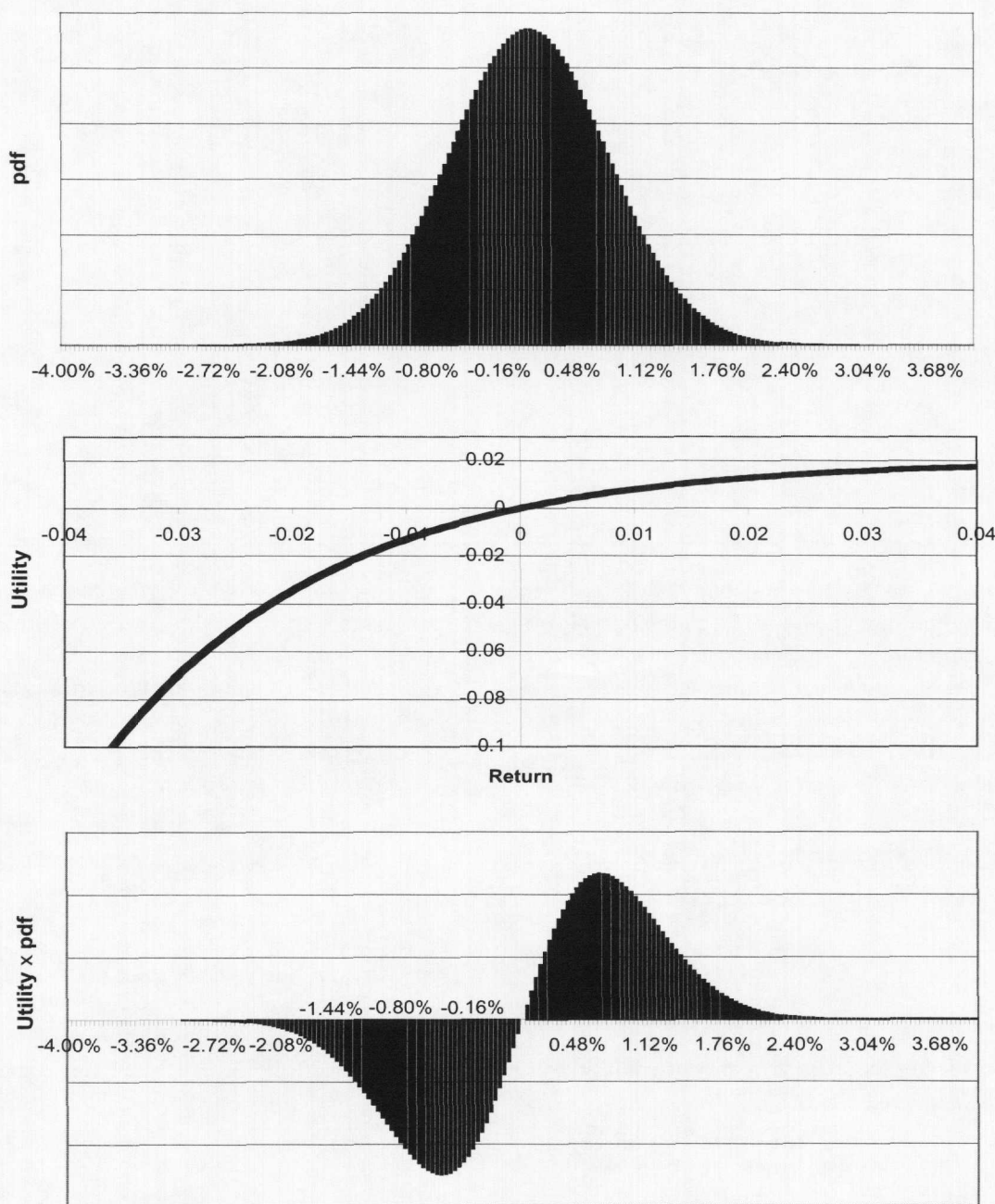
$$E(u_\pi) = \sum_i pdf(r_{\pi_i})u(r_{\pi_i}) \quad (7)$$

There is a unique portfolio—an optimal portfolio—that produces the highest utility. This can be found using, for example, Excel Solver.¹⁰ We can even take a shortcut, if we use the utility function described above and if risk is normally distributed. The formula is:¹¹

$$E(u) \cong \bar{r} - \lambda \sigma^2 \quad (8)$$

Thus, the optimal portfolio, which maximizes utility, is the same one that maximizes the expected return minus λ times the variance. This is exactly what Markowitz [1952] analyzes for varying levels of the risk aversion coefficient to trace out the efficient portfolio frontier.

**EXHIBIT 9
CALCULATION OF UTILITY**



This process is illustrated in Exhibit 9. The top panel shows the probability distribution function. The second panel shows our utility function. The bottom section is the result of multiplying the probability of each slice in the top panel by the utility of that excess return to result in the contribution to our expected utility.

One of the critical assumptions of this shortcut optimization is that, unlike scenario analysis, the probability of returns can only be normally distributed. If we want to work with non-normal distributions, the solution is to simply abandon the shortcut in favor of a direct approach. Using a direct approach, we can expressly

**EXHIBIT 10
COMBINING PROBABILITY DISTRIBUTIONS^a**

	Scenario 1	Scenario 2	Scenario 3	Sum
Probability of 1% Excess Return ^b	0.00%	1.33%	0.86%	
Probability of the Scenario	10%	70%	20%	
Joint Probability	0.00%	0.93%	0.17%	1.1%

^aThe probabilities are calculated for a portfolio that has a 25% underweight in both Treasuries and mortgages and a 50% overweight in corporates. Compared to Exhibit 1, the standard deviations of the spreads in each scenario are reduced so that the overall spread volatility is unchanged. This is achieved by reducing the volatility of mortgage and corporate spreads to Treasuries to 24 and 15 basis points, respectively.

^bThat is, a return between 0.99% and 1.01%.

combine the distributions from each of the scenarios so that we have a probability distribution that captures what may be non-normal expectations about the future. A simple example is in Exhibit 10.

Let's work with a 1% excess return. In scenario 1, where spreads are assumed to widen, the probability of this return is almost zero. In scenarios 2 and 3, the probabilities are much higher at 1.33% and 0.86%.¹² When we weight these probabilities by the scenario probabilities, we get the joint probability. The sum of the joint probabilities—1.1%—is the combined probability of a 1% excess return.

Continuing with this process, we calculate the probability-weighted returns across all the scenarios. This distribution is graphed in Exhibit 11. As you can see, the distribution is clearly not normal. We can then find the expected utility by taking the overall probability distribution for a given portfolio combination and multiplying it by our utility function. Using Excel Solver, we can repeat this process, until the combination is found that achieves the maximum utility.

In effect, what we have done conforms with a standard optimization process except that the probability distribution incorporates our scenarios and need not be normal. Now let's see what happens when we run our reengineered optimizer across a range of spreads.

Note in Exhibit 12 that there is surprisingly little difference from our first results (Exhibit 3), which employ a normal probability distribution. In both cases, the optimal portfolio holds no Treasuries when spreads range above very narrow levels. It requires somewhat greater inducement before mortgages are substantially underweighted with scenario optimization.

REEXAMINING OUR UTILITY FUNCTION

So far, we have addressed only one of the two major components of the optimization process: the probability distribution. Let's turn our attention to the other piece of the puzzle, the utility function. How can

**EXHIBIT 11
COMBINING SCENARIOS INTO A SINGLE NON-NORMAL DISTRIBUTION**

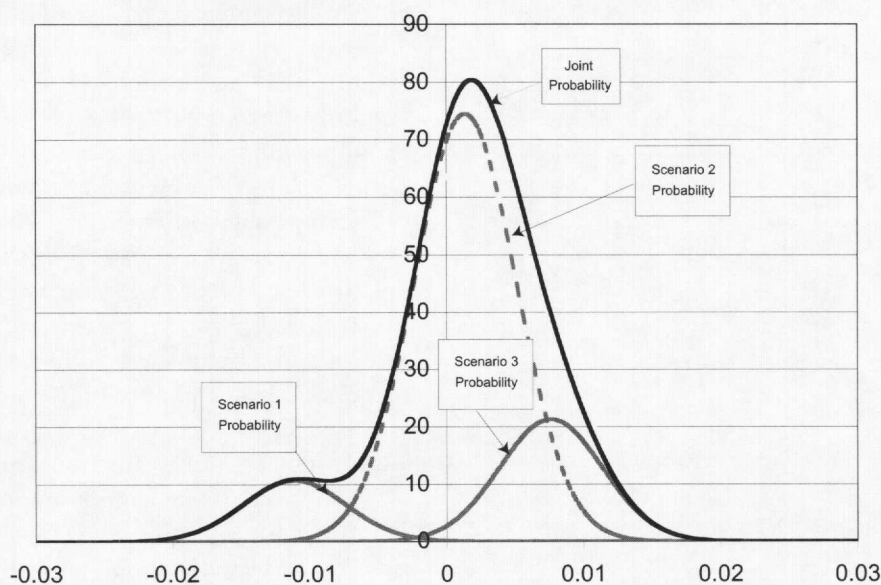
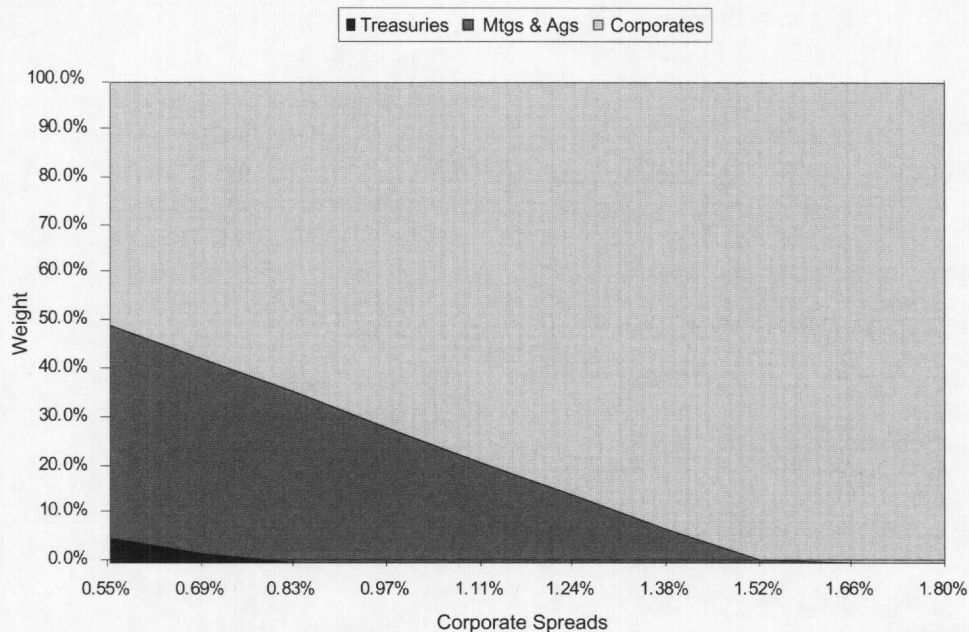


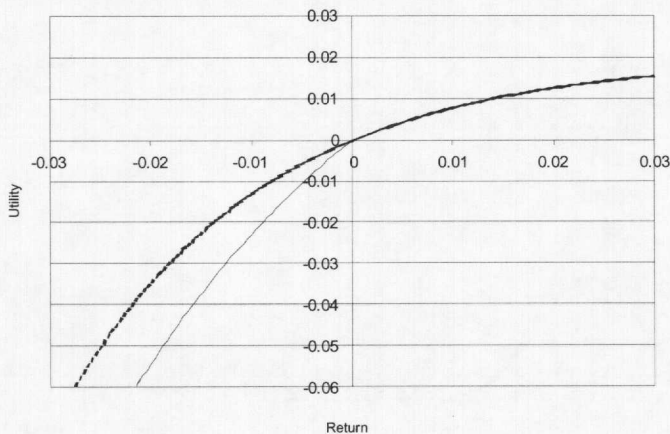
EXHIBIT 12
SECTOR WEIGHTS ACROSS SPREADS UNDER SCENARIO OPTIMIZATION



we be sure that it matches our attitudes toward risk and return?

There is no formulaic solution to this question. A utility function is meant to express the value to us of different outcomes. It may have a simple form like that given above, or it may not. It is well-documented that crossing the threshold between outperformance and underperformance qualitatively changes the way people feel. This has been termed “loss aversion.”¹³

EXHIBIT 13
NEW UTILITY FUNCTION WITH LOSS AVERSION



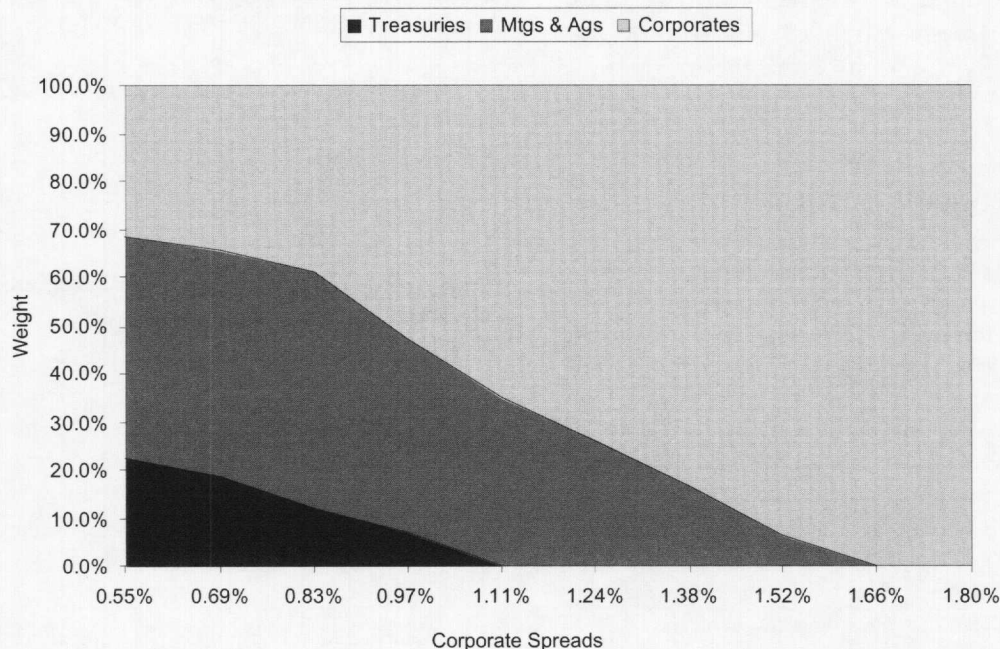
To reflect this, we introduce a kink in the utility function when it crosses zero, as shown by the solid line in Exhibit 13. The new utility function is much more averse to large losses. This should prevent us from creating a portfolio that has a meaningful probability of experiencing sizable losses relative to its benchmark—the utility function employed in a simple mean-variance optimization model would not prevent the creation of such a portfolio.¹⁴

Naturally, our new utility function is much more sensitive to the significant chance of substantial underperformance of corporates in the scenario distribution. When we optimize with this function, we arrive at the recommended weightings shown in Exhibit 14. To separate the effect of the utility function from that of the distribution, we use the normal distribution for these optimizations, so Exhibit 14 is comparable to Exhibit 3.

As you can see, this utility function maintains a Treasury weight even at much wider spreads. At the widest spreads, however, such as we experienced at year-end 2000, the optimal portfolio remains fully invested in corporates. Is this a better utility function? If this investment profile accords with our intuition, then we have found a suitable utility function. But if it doesn't, it isn't.

The point is that it is our investment preferences that determine our portfolio. The utility function simply provides the implementation.

EXHIBIT 14 SECTOR WEIGHTS ACROSS RANGE OF SPREADS



AREAS FOR FURTHER WORK

A related but more complex question is suggested by a problem we state as follows. Suppose we have perceived substantial opportunities and, accordingly, have taken a significant amount of risk. The market moves against us, creating serious but still acceptable losses. We believe that the original investment thesis is still sound and that the losses will be more than recouped, but if we continue to take substantial risk, we are at peril of entering an unacceptable loss range. What should we do in this case?

Ideally, the initial portfolio allocation should consider this dilemma. The solution requires a dynamic approach to the optimization. Such an approach would recognize the probability of sequential adverse events at the time of the initial allocation.¹⁵

CONCLUSION

The mean-variance optimization process relies on two implicit assumptions: first, that returns are normally distributed; and second, that our utility function has constant relative risk aversion. Neither of these assumptions may actually hold. As we have demonstrated, though, we can relax these assumptions by specifying the actual return distribution expected and by

creating a utility function that more closely captures our loss aversion preferences.

In the practical case of sector allocation in bond portfolios, altering the distribution alone has relatively little impact on portfolio construction. Treasuries still disappear from the portfolio at moderate spreads. Utility functions that apply a considerable penalty to large losses make the choice of optimal portfolios much more sensitive to adverse scenarios, however. The combination of a non-normal probability distribution and loss-averse utility function results in an optimization process that retains Treasuries in the portfolio even at yield spreads well above historical averages.

We think it is important for investors to consider whether the implicit assumptions of mean-variance optimization are accurate in their circumstances. The specification of the form of the utility function can be particularly crucial. In the particular case of fixed-income sector allocation, we would reduce the strong bias of mean-variance optimization toward corporates only slightly. We have found that quite large deviations from normality have little effect. While we do think it is appropriate to alter the utility function, we would not do so by as much as is indicated in the analysis. It may seem counter to the efficient markets theory that Treasuries would effectively be dominated by another invest-

ment, but we believe Treasuries are prized for their liquidity and certainty of payment. Therefore, it is rational for investors willing to sacrifice return for those attributes to own Treasuries while those who do not have special liquidity needs will avoid them.

APPENDIX A DOES THE BENCHMARK MATTER?

The way we have presented spreads, risks, and correlations, such as quoting spreads to Treasuries, appears to give Treasuries a special role in this analysis. One might question whether this is appropriate, given the reduction in Treasuries outstanding (because of the surplus) and the related increase in the use of swaps to hedge other sectors of the market.

We are sympathetic to the view that these developments may have altered the structure of the market. The central role of Treasuries in this analysis is merely a convenience, however; it is not essential. We could equally well have characterized the risk structure as in Exhibit A-1. This represents exactly the same risk structure but makes mortgages/agencies central.

To emphasize the fact that no sector or asset class plays a special role, our preference is to express the risk structure as risks between each pair of sectors as in Exhibit A-2.

EXHIBIT A-1 OPTIMIZATION PARAMETERS RESTATED VERSUS MORTGAGES/AGENCIES

	Spreads versus Mortgages/Agencies	Volatility	Correlation
Treasuries	-0.85%	0.33%	-0.2
Corporates	0.85%	0.25%	

EXHIBIT A-2 SPREAD RISKS EXPRESSED PAIRWISE

	Treasuries	Mortgages/Agencies
Treasuries	—	0.33%
Corporates	0.42%	0.25%

APPENDIX B CONSTANT ELASTICITY OF SUBSTITUTION UTILITY, NORMALITY, AND MEAN-VARIANCE OPTIMIZATION

Given the constant elasticity of substitution (CES) utility function:

$$u(r) = \frac{e^{-2\lambda r} - 1}{-2\lambda}$$

the expected utility assuming a normal distribution for r can be calculated from

$$E[u(r)] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \frac{(e^r)^{-2\lambda} - 1}{-2\lambda} e^{-\frac{(r-\bar{r})^2}{2\sigma^2}} dr$$

$$E[u(r)] = \frac{e^{-2\lambda(\bar{r} - \lambda\sigma^2)} - 1}{-2\lambda}$$

Hence maximizing $E[u(r)]$ is equivalent to maximizing $(\bar{r} - \lambda\sigma^2)$, as the former is monotonically increasing in the latter.

ENDNOTES

¹We calculate spreads using Treasuries as our reference point. We recognize that swaps are becoming more important in the calculation of spreads, but for our purposes, the difference is irrelevant. The logic of this assertion is presented in Appendix A.

²The multiple 25 reflects a typical level of risk aversion for this problem. Roughly, it means that, in a portfolio with 100 basis points of tracking error, we are willing to take 2 basis points of additional tracking error, provided it increases the expected excess return by 1 basis point.

³Source: Plan Sponsor Network.

⁴We have also experimented with different choices of risk aversion parameter. The substantial underweight of Treasuries persists even for relatively high degrees of risk aversion.

⁵The cost of defaults to A-rated corporate bonds averaged 10 basis points between 1970 and 1994. See Carty [1995].

⁶See, for example, Ait-Sahalia and Brandt [2001], who express it differently but use values corresponding to 0.5, 2, 4.5, and 9.5.

⁷All assets maintain the same expected return and standard deviations as in the original assumptions. Because we want to change the assumption of normality while maintaining the same expected returns and variances, we have engineered probabilities that counterbalance the skew in the spread changes.

⁸A further justification for the mean-variance framework and a description of the conditions under which it is truly optimal are in Merton [1969] and Samuelson [1969]. Notable work relating to its limitations and its alternatives includes Epstein and Zin [1991], who propose an alternative utility specification; Ang, Bekaert, and Liu [2000], who propose the utility function as an explanation for the equity premium puzzle; and Koskosidis and Duarte [1997], who discuss scenario-based optimization. Campbell and Viceira [2000] also provide a thorough discussion of this topic.

⁹Technically, this utility specification is known as the constant elasticity of substitution (CES) utility function, or constant relative risk aversion. This utility function is consistent with the mean-variance optimization process with normally distributed returns. It should also be noted that we consider only relative return. In some cases, it is important to consider both tracking error and absolute risk, and this utility function does not do so. This can be incorporated into the framework, but we do not do so here for reasons of simplicity.

¹⁰For problems with many independent risks, this is intractable even with current computing power. Fortunately, this is when scenario optimization is least important. If one is subject to a great number of separate risks, diversification is likely to result in a distribution that is roughly normal. When there are fewer separate risks—as is common in fixed-income portfolios—probabilities can depart most from normality. These cases are computationally feasible.

¹¹The derivation of Equation (8) is in Appendix B.

¹²The probability of a 1% return is lower in scenario 3 than in scenario 2 because the mean of the distribution (for scenario 3) is substantially above 1%.

¹³Numerous studies support our experience. See, for example, Kahneman and Tversky [1984].

¹⁴The new utility function is arrived at by simply subtracting -1 times the return from the original CES utility function whenever the return is negative (something we invented to capture loss aversion).

¹⁵The importance of an intertemporal approach was recognized by Merton [1969] and Samuelson [1969] but was computationally intractable at that time. Recent work has been done by Barberis [2000], Brennan, Schwartz, and Lagnado [1997], and Campbell, Chan, and Viceira [2000].

ENDNOTE

The authors thank William Marshall for his comments.

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